MATH 3060 Tutorial 2

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Remarks on Homework 1:

(i) Please match your answer with the corresponding question when you submit the homework

(ii) When you write down a Fourier series, don't forget the $\sin nx, \cos nx$.

(iii) A Lipschitz continuous function need not be differentiable

(iv) I don't know why many of you wrote $\sin\sqrt{2n\pi + \pi/2} = 1$.

1. Determine whether the followings are true or not.

   (a) If f is differentiable and f' is bounded on $[0,1]$, then f is uniform Lipschitz on $[0,1]$

   (b) If is Lipschitz on $[0,1]$, and f is differentiable, then f' is bounded on $[0,1]$.

   (c) The function $f(x) = x^2$ is uniformly Lipschitz on $[0,1]$.

   (d) There exists no integrable functions f on $[-\pi,\pi]$ so that

   $$f \sim \sum_{n=1}^{\infty} \sin nx.$$

   (e) There exists no integrable functions f on $[-\pi,\pi]$ so that

   $$f \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos nx.$$

   (f) Let $f_n \to f$ on $[0,1]$ in L^2 sense, then $f_n(x) \to f(x)$ for some $x \in [0,1]$.

   (g) If $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly (i.e. the partial sum $s_N = \sum_{n=-N}^{N}$ converges uniformly), then $\sum_{n=-\infty}^{\infty} \frac{1}{|c_n|^2} < \infty$.

   (h) If $\sum_{n=-\infty}^{\infty} \frac{1}{|c_n|^2} < \infty$, then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly.

   (i) Let $c_n = c_n(f)$ for some function f integrable on $[-\pi,\pi]$, then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges for almost all $x \in [-\pi,\pi]$.

(j) Let f be a 2π periodic continuous, suppose $c_n(f) = 0$ for all n. Then f is the zero function.

2. Let $0 < \delta < \pi$, and define the 2π periodic function f by

$$f(x) = \begin{cases} 1, & \textit{if } |x| \leq \delta \\ 0, & \textit{if } \delta < |x| \leq |\pi| \end{cases}$$

(a) Compute the Fourier coefficients of f.

(b) Show that

$$\sum_{n=1}^{\infty} \frac{\sin n\delta}{n} = \frac{\pi - \delta}{2}.$$

(c) Show that

$$\sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2} = \frac{\pi - \delta}{2}.$$

(d) Show that

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$